Global invariants of paths and curves for the group of orthogonal transformations in the two-dimensional Euclidean space

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Abstract

In this paper, for the orthogonal group $G = O(2)$ and special orthogonal group $G = O^+(2)$ global G-invariants of plane paths and plane curves in two-dimensional Euclidean space E_2 are studied. Using complex numbers, a method to detect G-equivalences of plane paths in terms of the global G-invariants of a plane path is presented. General evident form of a plane path with the given G-invariants are obtained. For given two plane paths $x(t)$ and $y(t)$ with the common G-invariants, evident forms of all transformations $g \in G$, carrying $x(t)$ to $y(t)$, are obtained. Similar results have obtained for plane curves.

1 Introduction

Transformations and invariants of curves, surfaces and graphical objects appear in many areas of Computer Aided Geometric Design, Computer Aided Design, computer graphics, computer vision and pattern recognition. One of the most important problems in computer vision and vision-based applications is the automatic identification of objects. Plane curves are fundamental objects in differential geometry. In industrial shape design or computer aided geometric design, curves of particular property have been used as a parts of

figures (see in some references [1, 6, 7, 10, 11, 18] and therein references). The important problem is to find simple but efficient method for the equivalence check of two curves. The present paper is devoted to the global differential geometry of paths and curves in two-dimensional Euclidean space E_2 for the two-dimensional orthogonal group $O(2)$ and the special orthogonal group $O^+(2) = \{g \in O(2) | det(g) = 1\}$.

Below we use following definitions of a path, an L-regular path, a curve, a G-equivalence of paths and a G-equivalence of curves in the 2-dimensional Euclidean space E_2.

Let R be the field of real numbers and $T = (a, b)$ be an open interval in R.

Definition 1.1. A $C^{(1)}$-mapping $x : T \to E_2$ will be called a T-path (parametrized curve) in E_2.

Let $G = O(2)$ or $G = O^+(2)$.

Definition 1.2. T-paths $x(t)$ and $y(t)$ in E_2 are called G-equivalent if there exists $F \in G$ such that $y(t) = Fx(t)$. This being the case , we write $x(t) \overset{G}{\sim} y(t)$

Definition 1.3. A function $f(x(t), y(t), \dots, z(t))$ of a finite number of T-paths $x(t), y(t), \dots, z(t)$ is called G-invariant if $f(Fx(t), Fy(t), \dots, Fz(t)) = f(x(t), y(t), \dots, z(t))$ for all $F \in G$, for all T-paths $x(t), y(t), \dots, z(t)$ and for all $t \in T$.

Definition 1.4. A T-path $x(t)$ in E_2 will be called L-regular if $x(t) \neq 0$ for all $t \in T$.

Let $J_1 = (a, b) \subseteq R$ and $J_2 = (c, d) \subseteq R$.

Definition 1.5. (see [4, 15]) A J_1-path $x(t)$ and a J_2-path $y(r)$ in E_2 will be called D-equivalent if a $C^{(1)}$-diffeomorphism $\varphi : J_2 \to J_1$ exists such that $\varphi'(r) > 0$ and $y(r) = x(\varphi(r))$ for all $r \in J_2$. A class of D-equivalent paths in E_2 will be called a curve (non-parametrized curve) in E_2. A path $x \in \alpha$ will be called a parametrization of a curve α.

Let $\alpha = \{h_\tau, \tau \in \Gamma\}$ be a curve in E_2, where h_τ is a parametrization of α. Then $F\alpha = \{Fh_\tau, \tau \in \Gamma\}$ is a curve in E_2 for any $F \in O(2)$.

Definition 1.6. (see [4, 15]) Two curves α and β in E_2 are called G-equivalent if $\beta = F\alpha$ for some $F \in G$. This being the case , we write $\alpha \overset{G}{\sim} \beta$.

Definition 1.7. A curve α will be called L-regular if it contains an L-regular path.

Remark 1.8. There are some definitions of specific curves available, related to the subject of our research, given by [2], [9, p.2], [12, p.25], [17, p.20-22].

For *regular* curves in classic differential geometry, the following theorem is known for the group $M^+(2)$: "Let α and β be two curves in the two-dimensional Euclidean space E_2. Then, α and β are $M^+(2)$-equivalent if and only if the curvatures and speeds of α and β are equal", where $M^+(2)$={$F : E_2 \to E_2 \mid Fx = gx + b,\ g \in O^+(2),\ b \in E_2$}.

This theorem shows that a necessary and sufficient condition for two curves in E_2 to be $M^+(2)$-equivalent is that they have the same curvature and speed. Using method of Frenet frames of two curves, this theorem shows how to compute explicitly an isometry carrying α to β (see [19]). In addition, the method give conditions only for *the local $M^+(2)$-equivalence* of curves.

Generally, in the classical differential geometry of curves in the n-dimensional Euclidean space E_n, where $n \geq 3$, using Frenet-Serret equations, curvature functions $k_i, i = 1, \ldots, n - 1$, of the curve were obtained (see [3, p.172]). The function k_i is $M(n)$-invariant for $i = 1, \ldots n - 2$, where $M(n)$={$F : E_n \to E_n \mid Fx = gx + b,\ g \in O(n),\ b \in E_n$}, $O(n)$ is the group of all orthogonal transformations of E_n. But k_{n-1} is not $M(n)$-invariant. It is $M^+(n)$-invariant, where $M^+(n)$={$F : E_n \to E_n \mid Fx = gx + b,\ g \in O^+(n),\ b \in E_n$}, $O^+(n) = \{g \in O(n) \mid det(g) = 1\}$ is the group of all special orthogonal transformations of E_n. Thus invariant theory of curves in the classical differential geometry was developed only for the group $M^+(n)$. In addition, the method of orthogonal frame in the classical differential geometry give conditions only for *the local $M^+(n)$-equivalence* of curves.

In works [4, 12], by using invariant parametrization of curves, the problem of global G-equivalence of curves (that is nonparametrized curves) was reduced to the problem of global G-equivalence of paths (that is parametrized curves) for groups $G = M(n), M^+(n)$. Complete systems of global invariants of *non-degenerate* paths and *non-degenerate* curves in classical geometries were obtained. Moreover, systems of generators of differential field of G-invariant differential rational functions of a path in E_n for groups $M(n)$ and $M^+(n)$ are obtained . This approach was developed for curves in papers [15, 16, 21, 22] and for vector fields in [13, 14].

In paper [20], an invariant-theoretic approach to problems of G-equivalence of Bézier curves for groups $G = M^+(n), M(n)$ is given. These papers contain solutions of problems of global G-equivalence of Bézier curves for these groups without using differential invariants of a Bézier curve.

The present paper is devoted to an investigation of problems of global G-equivalence of *regular* and *non-degenerate* paths in E_2 for groups $G = O(2), O^+(2)$ in terms of global G-invariants of paths. For *non-degenerate* curves, this theory is also investigated. For solutions of these problems, we use complex numbers.

In the book [8, Theorems 6.1, 6.8], for the group $M^+(2)$, existence and

uniqueness theorems for regular parametric curves in E_2 were obtained by using complex numbers. In the book [24], different concrete classes of plane curves investigated by using complex numbers. In the end of this book, "Historical notes" on the development of the concept of the complex number and applications of complex numbers are given. Some applications of three types of general complex numbers (ordinary complex numbers, hyperbolic numbers and dual numbers) to two-dimensional geometries are given in the book [23].

This paper is organized as follows. In Sec. 2, some known results on descriptions of groups $G = O(2)$ and $O^+(2)$ in terms of complex numbers are given. The fundamental G-invariants of a path in E_2 for the groups $G = O(2), O^+(2)$ are given. In Sec. 3, complete systems of global G-invariants of a regular path and uniqueness theorems for regular paths are given. In Sec. 4, theorems on an existence for regular paths and evident forms of a path with given G-invariants are obtained. In Sec. 5, the type and invariant parametrizations of an L-non-degenerate curve are defined and investigated. In Sec. 6, complete systems of G-invariants of an L-non-degenerate curve and uniqueness theorems for L-non-degenerate curves are obtained.

2 The groups $G = O(2), O^+(2)$ and fundamental G-invariants of a path in E_2

Let R be the field of real numbers and Ω be the field of complex numbers. The multiplication in Ω has the form $(a_1 + ia_2)(b_1 + ib_2) = (a_1b_1 - a_2b_2) + i(a_1b_2 + a_2b_1)$. We will consider element $a = a_1 + ia_2$ also in the form $a = \begin{pmatrix} a_1 \\ a_2 \end{pmatrix}$.

For $a = a_1 + ia_2$, denote by P_a the matrix $\begin{pmatrix} a_1 & -a_2 \\ a_2 & a_1 \end{pmatrix}$ and consider P_a also as the transformation $P_a : \Omega \to \Omega$, where $P_a b = \begin{pmatrix} a_1 & -a_2 \\ a_2 & a_1 \end{pmatrix} \begin{pmatrix} b_1 \\ b_2 \end{pmatrix} = \begin{pmatrix} a_1b_1 - a_2b_2 \\ a_1b_2 + a_2b_1 \end{pmatrix}$ for $b = b_1 + ib_2 = \begin{pmatrix} b_1 \\ b_2 \end{pmatrix}$. Then we have the equality

$$ab = P_a b. \tag{1}$$

for all $a, b \in \Omega$. Let $P(\Omega)$ denote the set of all matrices P_a, where $a \in \Omega$. We consider on $P(\Omega)$ the following standard matrix operations: the componentwise addition, a scalar multiplication and the multiplication of matrices. Then $P(\Omega)$ is a field, where the unit element is the unit matrix. The following Propositions 2.1-2.5 are known.

Proposition 2.1. *The mapping $P : \Omega \to P(\Omega)$, where $P : a \to P_a$ for all $a \in \Omega$, is an isomorphism of fields.*

For vectors $a = a_1 + ia_2, b = b_1 + ib_2 \in \Omega$, we put $< a, b >= a_1b_1 + a_2b_2$. Then $< a, b >$ is a bilinear form on E_2 and $< a, a >= a_1^2 + a_2^2$ is a quadratic form on E_2. Put $Q(a) =< a, a >$. We consider the field Ω also as the two-dimensional Euclidean space E_2 with the scalar product $< a, b >$. Then $\|a\| = |a| = \sqrt{Q(a)}, \forall a \in \Omega$.

Proposition 2.2.   *(i) Equalities $Q(a) = det(P_a)$, $Q(ab) = Q(a)Q(b)$, $|ab| = |a| \, |b|$, $Q(a) = det(P_a)$ hold for all $a, b \in \Omega$.*

(ii) Let $a = a_1 + ia_2 \in \Omega^*$. Then $det(P_a) = Q(a) = a_1^2 + a_2^2 > 0$.

An endomorphism ψ of a vector space Ω is called an involution of the field Ω if $\psi(\psi(a)) = a$ and $\psi(ab) = \psi(a)\psi(b)$ for all $a, b \in \Omega$. For an element $a = a_1 + ia_2 \in \Omega$, we set $\overline{a} = a_1 - ia_2$.

Proposition 2.3. *The mapping $a \to \overline{a}$ is an involution of the field Ω. In addition, for an arbitrary element $a = a_1 + ia_2 \in \Omega$, equalities $a + \overline{a} = 2a_1, < a, a >= a\overline{a} = a_1^2 + a_2^2 \in R$ hold.*

Proposition 2.4. *Let $a \in \Omega$. Then the element a^{-1} exists if and only if $Q(a) \neq 0$. In the case $Q(a) \neq 0$, equalities $a^{-1} = \frac{\overline{a}}{Q(a)}$ and $Q(a^{-1}) = \frac{1}{Q(a)}$ hold.*

Let $W = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}$. We will use W also for the writing of the element $\overline{z}$ in the form $\overline{z} = Wz$.

Proposition 2.5. $Q(Wa) = Q(a)$ *for all $a \in \Omega$ and $< Wa, Wb >=< a, b >$ for all $a, b \in \Omega$.*

Let $a = a_1 + ia_2 \in \Omega^*$ that is $|a| \neq 0$. Put

$$P_a^+ = \begin{pmatrix} \frac{a_1}{|a|} & \frac{-a_2}{|a|} \\ \frac{a_2}{|a|} & \frac{a_1}{|a|} \end{pmatrix}.$$

Proposition 2.6. *Let $a = a_1 + ia_2 \in \Omega^*$. Then the equality $P_a = |a| \, P_a^+$ holds, where $P_a^+ \in O^+(2)$.*

Proof. The equality $P_a = |a| \, P_a^+$ is obvious. Since $(\frac{a_1}{|a|})^2 + (\frac{a_2}{|a|})^2 = 1$, the implication $P_a^+ \in O^+(2)$ follows from [5, p.161-162]. $\square$

Put $\Omega^* = \{z \in \Omega \mid Q(z) \neq 0\}$. Ω^* is a group with respect to the multiplication operation in the field Ω. Put $S(\Omega^*) = \{z \in \Omega \mid Q(z) = 1\}$, $P(\Omega^*) = \{P_z \mid z \in \Omega^*\}$ and $P(S(\Omega^*)) = \{P_z \mid z \in S(\Omega^*)\}$. $S(\Omega^*)$ is a subgroup of the

group Ω^* and $S(\Omega^*) = \left\{ e^{i\varphi} \mid \varphi \in R \right\}$. Denote by $P(\Omega^*)W$ the set of all matrices $\{gW \mid g \in P(\Omega^*)\}$, where gW is the multiplication of matrices g and W.

We note that $O^+(2) \cap O^-(2) = \emptyset$ and the set $O^+(2)$ is a subgroup of $O(2)$.

Theorem 2.7. *The following equalities are hold:*

(i) $O^+(2) = \{F : E_2 \to E_2 | F(x) = P_a x, a \in S(\Omega^*), \forall x \in E_2\}$

(ii) $O^-(2) = \{F : E_2 \to E_2 | F(x) = P_a W(x), a \in S(\Omega^*), \forall x \in E_2\}$

(iii) $O(2) = O^+(2) \cup O^-(2)$.

Remark 2.8. It is obvious that a T-path $x(t)$ in E_2 is L-regular if and only if $Q(x(t)) \neq 0$ for all $t \in T$.

Let $x(t) = (x_1(t), x_2(t))$ be a T-path in E_2 and $x'(t) = (x'_1(t), x'_2(t))$ is its first derivative. Denote the determinant of vectors $x(t), x'(t)$ by $[x(t)\, x'(t)]$.

The following proposition is known.

Proposition 2.9.   (i) *The functions* $Q(x(t)) =< x(t), x(t) >, < x(t), x'(t) >$
and $< x'(t), x'(t) >$ *are* $O(2)$-*invariant on the set of all* L-*regular* T-*paths* $x(t)$ *in* E_2;

(ii) *The function* $\frac{<x(t),x'(t)>}{Q(x(t))}$ *is* $O(2)$-*invariant on the set of all* L-*regular* T-*paths* $x(t)$ *in* E_2;

(iii) *The function* $[x(t)\, x'(t)]$ *is* $O^+(2)$-*invariant on the set of all* L-*regular* T-*paths* $x(t)$ *in* E_2;

(iV) *The function* $\frac{[x(t)\, x'(t)]}{Q(x(t))}$ *is* $O^+(2)$-*invariant on the set of all* L-*regular* T-*paths* $x(t)$ *in* E_2;

(V) *The function* $\frac{[x(t)\, x'(t)]^2}{Q^2(x(t))}$ *is* $O(2)$-*invariant on the set of all* L-*regular* T-*paths* $x(t)$ *in* E_2;

(Vi) *The function* $\frac{<x'(t),x'(t)>}{Q(x(t))}$ *is* $O(2)$-*invariant on the set of all* L-*regular* T-*paths* $x(t)$ *in* E_2.

3 Complete systems of invariants of an L-regular path and uniqueness theorems for L-regular paths

Proposition 3.1. *Let* $u, v \in \Omega$. *Assume that* $Q(u) \neq 0$. *Then the element* vu^{-1} *exists, the following equalities hold:*

$$vu^{-1} = \frac{< u, v >}{Q(u)} + i\frac{[u\,v]}{Q(u)}$$

and

$$P_{vu^{-1}} = \begin{pmatrix} \frac{<u,v>}{Q(u)} & -\frac{[u\,v]}{Q(u)} \\ \frac{[u\,v]}{Q(u)} & \frac{<u,v>}{Q(u)} \end{pmatrix}. \tag{2}$$

Proof. Let $u = u_1 + iu_2, v = v_1 + iv_2$. Since $Q(u) \neq 0$, element u^{-1} exists by Proposition 2.4. Hence vu^{-1} exists. By Proposition 2.4, $u^{-1} = \frac{\overline{u}}{Q(u)}$. Using $\overline{u} = u_1 - iu_2$ and the multiplication in the field Ω, we obtain the equality $vu^{-1} = \frac{<u,v>}{Q(u)} + i\frac{[u\,v]}{Q(u)}$. This equality and the notation of P_a imply (2). $\square$

Theorem 3.2. *(i) Let* $x(t)$ *and* $y(t)$ *be L-regular T-paths in* E_2 *such that* $x(t) \overset{O^+(2)}{\sim} y(t)$. *Then, for all* $t \in T$, *the following equalities hold*

$$\begin{cases} < x(t), x(t) >= < y(t), y(t) >, \\ [x(t)\,x'(t)] = [y(t)\,y'(t)]. \end{cases} \tag{3}$$

(ii) Conversely, if $x(t)$ *and* $y(t)$ *are L-regular T-paths in* E_2 *such that the equalities (3) hold, then* $x(t) \overset{O^+(2)}{\sim} y(t)$. *Moreover, there exists the unique* $F \in O^+(2)$ *such that* $y(t) = Fx(t)$ *for all* $t \in T$. *In this case,* $F = U$, *where* $U \in O^+(2)$, U *has the following form*

$$U = \begin{pmatrix} \frac{<x(t),y(t)>}{Q(x(t))} & -\frac{[x(t)\,y(t)]}{Q(x(t))} \\ \frac{[x(t)\,y(t)]}{Q(x(t))} & \frac{<x(t),y(t)>}{Q(x(t))} \end{pmatrix}, \tag{4}$$

and the matrix U *does not depend on* $t \in T$.

Proof. (i) Assume that $x(t) \overset{O^+(2)}{\sim} y(t)$. Then $g \in O^+(2)$ exists such that $y(t) = gx(t)$ for all $t \in T$. Using this equality, we obtain $< y(t), y(t) >=< gx(t), gx(t) >=< x(t), x(t) >$ for all $t \in T$. Hence the equality $< x(t), x(t) >=< y(t), y(t) >$ holds for all $t \in T$. Using $det(g) = 1$, we obtain $[y(t)\,y'(t)] = [gx(t)\,gx'(t)] = det(g)\,[x(t)\,x'(t)] = [x(t)\,x'(t)]$. Hence the equality $[x(t)\,x'(t)] = [y(t)\,y'(t)]$ holds for all $t \in T$.

(ii) Conversely, assume that equalities (3) hold. Since $x(t)$ and $y(t)$ are L-regular paths, we have $Q(x(t)) \neq 0$ and $Q(y(t)) \neq 0$ for all $t \in T$. Then, by Proposition 2.4, $(x(t))^{-1}$ and $(y(t))^{-1}$ exist for all $t \in T$. We put $u = y(t), v = y'(t)$ in Proposition 3.1. Then we obtain

$$\frac{y'(t)}{y(t)} = \frac{< y(t), y'(t) > +i\,[y(t)\,y'(t)]}{Q(y(t))}. \tag{5}$$

Similarly, we obtain

$$\frac{x'(t)}{x(t)} = \frac{< x(t), x'(t) > +i\,[x(t)\,x'(t)]}{Q(x(t))}. \tag{6}$$

Equalities $< x(t), x'(t) >= \frac{1}{2}\frac{d}{dt} < x(t), x(t) >$, $< y(t), y'(t) >= \frac{1}{2}\frac{d}{dt} < y(t), y(t) >$ and the equality $< x(t), x(t) >=< y(t), y(t) >$ in the equalities (3) imply the equality $< x(t), x'(t) >=< y(t), y'(t) >$ for all $t \in T$. This equality, the equality $Q(x(t)) =< x(t), x(t) >=< y(t), y(t) >= Q(y(t))$, the equality $[x(t)\,x'(t)] = [y(t)\,y'(t)]$ in (3) and the equalities (5), (6) imply the equality $y'(t)(y(t))^{-1} = x'(t)(x(t))^{-1}$. Hence $y'(t)(y(t))^{-1} - x'(t)(x(t))^{-1} = 0$ for all $t \in T$. Using this equality, we obtain

$$\frac{d(y(t)(x(t))^{-1})}{dt} = y'(t)(x(t))^{-1} - y(t)x'(t)(x(t))^{-2} =$$
$$y(t)(y'(t)(y(t))^{-1} - x'(t)(x(t))^{-1})(x(t))^{-1} = 0$$

for all $t \in T$. This means that the function $y(t)(x(t))^{-1}$ is a constant on T. Put $g = y(t)(x(t))^{-1}$. Since $Q(x(t)) \neq 0$ and $Q(y(t)) \neq 0$ for all $t \in T$, we have $g \neq 0$.

Since $y(t) = y(t)(x(t))^{-1}x(t)) = (y(t)(x(t))^{-1})x(t)$, we have $y(t) = gx(t)$. Using (1), we obtain $y(t) = gx(t) = P_g x(t)$. The equality $< x(t), x(t) >=< y(t), y(t) >$ in (3) implies $< y(t), y(t) >=< P_g x(t), P_g x(t) >=< x(t), x(t) >=$ for all $t \in T$. Since P_g is linear, we obtain $P_g \in O(2)$. Then $det(P_g) = 1$ or $det(P_g) = -1$. Using the equality $[x(t)\,x'(t)] = [y(t)\,y'(t)]$ in the equalities (3), we obtain that $det(P_g) = 1$ and $P_g \in O^+(2)$. Hence $x(t) \overset{O^+(2)}{\sim} y(t)$.

By $g = y(t)(x(t))^{-1} = \frac{<x(t),y(t)>}{Q(x(t))} + i\frac{[x(t)\,y(t)]}{Q(x(t))}$ and Proposition 3.1, P_g has the form (4) and $P_g = U$. Since g is a constant, the matrix U does not depend on $t \in T$.

Prove the uniqueness of $U \in O^+(2)$ satisfying the condition $y(t) = Ux(t)$. Assume that $H \in O^+(2)$ exists such that $y(t) = Hx(t)$. Then by (1),

Proposition 2.1 and Theorem 2.7(i), there exists the unique $a \in \Omega^*$ such that $H = P_a$. Hence we have $y(t) = P_a x(t)$. By (1), we obtain $y(t) = ax(t)$. Since $Q(x(t)) \neq 0$, $y(t) = ax(t)$ implies that $a = y(t)(x(t))^{-1} = g$. Hence $P_a = P_g = U$. The uniqueness of U is proved.

$\square$

Example 3.3. Consider L-regular T-paths $x(t) = (\frac{t^4+1}{t^2+1}, \frac{t^2}{t^2+1})$ and $y(t) = (\frac{\sqrt{3}-t^2+\sqrt{3}t^4}{2+2t^2}, \frac{1+\sqrt{3}t^2+t^4}{2+2t^2})$, where $T = R$. It is easy to see that equalities in the equalities (3) hold for these paths $x(t)$ and $y(t)$. Then, by Theorem 3.2(ii), $x(t) \overset{O^+(2)}{\sim} y(t)$. Moreover, by Theorem 3.2(ii), we obtain that $U = \begin{pmatrix} \sqrt{3}/2 & -1/2 \\ 1/2 & \sqrt{3}/2 \end{pmatrix}$.

Definition 3.4. A T-path $x(t)$ in E_2 is called completely L-degenerate if $[x(t)\, x'(t)] = 0$ for all $t \in T$.
A T-path $x(t)$ in E_2 is called L-non-degenerate if $[x(t)\, x'(t)] \neq 0$ for all $t \in T$.

It is obvious that every L-non-degenerate path is L-regular. If $x(t)$ and $y(t)$ are T-paths in E_2 such that $x(t)$ is completely L-degenerate and $x \overset{O(2)}{\sim} y$, then $y(t)$ is also completely L-degenerate. Similarly, it is obvious that if $x(t)$ is L-non-degenerate and $x \overset{O(2)}{\sim} y$, then $y(t)$ is also L-non-degenerate.

Example 3.5. Consider the T-path $x(t) = (t^2, \frac{t^2}{2})$ in E_2, where $T = (0,1)$. Then, $[x(t)\, x'(t)] = 0$ for all $t \in T$. That is, $x(t)$ is a completely L-degenerate T-path. Since $t^2 \neq 0$ for all $t \in T$, we obtain that $x(t) = (t^2, \frac{t^2}{2})$ is a completely L-degenerate L-regular T-path.

Theorem 3.6. (i) Let $x(t), y(t)$ be completely L-degenerate L-regular T-paths in E_2 such that $x(t) \overset{O(2)}{\sim} y(t)$. Then, for all $t \in T$, the following equality holds:

$$< x(t), x(t) > = < y(t), y(t) > . \qquad (7)$$

(ii) Conversely, if $x(t)$ and $y(t)$ are completely L-degenerate L-regular T-paths in E_2 such that the equality (7) holds, then $x(t) \overset{O(2)}{\sim} y(t)$. Moreover, only two elements $F = F_1, F_2 \in O(2)$ exist such that $y(t) = F(x(t))$ for all $t \in T$. Here $F_1 = U_1, F_2 = (U_2 W)$, where $U_1, U_2 \in O^+(2)$, the matrix U_1 has the form (4) and the matrix U_2 has the form

$$U_2 = \begin{pmatrix} \frac{<Wx(t),y(t)>}{Q(Wx(t))} & -\frac{[Wx(t)\,y(t)]}{Q(Wx(t))} \\ \frac{[Wx(t)\,y(t)]}{Q(Wx(t))} & \frac{<Wx(t),y(t)>}{Q(Wx(t))} \end{pmatrix} . \qquad (8)$$

The matrices U_1 and U_2 are independent from variable on $t \in T$.

Proof.   (i) Assume that $x(t) \overset{O(2)}{\sim} y(t)$. The function $< x(t), x(t) >$ is $O(2)$-invariant, so the equality (7) holds.

(ii) Conversely, assume that equality (7) holds. Since $x(t)$ and $y(t)$ are completely L-degenerate paths, we have

$$[x(t)\, x'(t)] = [y(t)\, y'(t)] = 0. \qquad (9)$$

for all $t \in T$. This equality and the equality (7) imply the equalities (3). Then, by Theorem 3.2(*ii*), the unique $F \in O^+(2)$ exists such that $y(t) = Fx(t)$ for all $t \in T$. This means that the unique $U_1 \in O^+(2)$ exists such that $y(t) = U_1 x(t)$ for all $t \in T$. Here, the matrix U_1 has the form (4). By Theorem 3.2, the matrix U_1 does not depend on $t \in T$.

Consider the T-path $Wx(t)$. Since the function $< x(t), x(t) >$ is W-invariant, we have $< Wx(t), Wx(t) >=< x(t), x(t) >$ for all $t \in T$. Similarly we obtain $< Wy(t), Wy(t) >=< y(t), y(t) >$ for all $t \in T$. Using $det(W) = -1$ and the equality (9) for all $t \in T$, we obtain

$$\left[Wx(t)\, (Wx)^{'}(t) \right] = det(W)\, [x(t)\, x'(t)] = -\, [x(t)\, x'(t)] = [y(t)\, y'(t)] = 0$$

Using the equalities $< Wx(t), Wx(t) >=< x(t), x(t) >$ and $\left[(Wx(t))\, (Wx)^{'}(t)) \right] = [y(t)\, y'(t)] = 0$, we obtain equalities:

$$\begin{cases} < Wx(t), Wx(t) >= < y(t), y(t) >, \\ \left[Wx(t)\, (Wx)^{'}(t) \right] = [y(t)\, y'(t)] . \end{cases}$$

for all $t \in T$. Then, by Theorem 3.2(*ii*), the unique $U_2 \in O^+(2)$ exists such that $y(t) = U_2(Wx(t)) = (U_2 W)x(t)$ for all $t \in T$. Here, the matrix U_2 has the form (8). By Theorem 3.2, the matrix U_2 does not depend on $t \in T$.

Let $F \in O(2)$ such that $y(t) = Fx(t)$. Prove that $Fx(t) = U_1 x(t)$ or $Fx(t) = (U_2 W)x(t)$. Let $y(t) = Fx(t) = Cx$ for some $C \in O(2)$. Then $C \in O^+(2)$ or $C \in O^-(2) = O^+(2)W$. Assume that $C \in O^+(2)$. Then, by the uniqueness in Theorem 3.2, $C = U_1$. Assume that $C \in O^+(2)W$. Then C has the form $C = DW$, where $D \in O^+(2)$. We have $y(t) = (DW)x(t) = D(Wx(t))$. Hence paths $y(t)$ and $Wx(t)$ are $O^+(2)$-equivalent. By the uniqueness in Theorem 3.2, $D = U_2$. $\qquad\square$

Theorem 3.7.   *(i) Let $x(t)$ and $y(t)$ be L-non-degenerate T-paths in E_2 such that $x \overset{O(2)}{\sim} y$. Then, for all $t \in T$, the following equalities hold*

$$\begin{cases} < x(t), x(t) >= < y(t), y(t) >, \\ [x(t)\, x'(t)]^2 = [y(t)\, y'(t)]^2 . \end{cases} \tag{10}$$

(ii) *Conversely, if $x(t)$ and $y(t)$ are L-non-degenerate T-paths in E_2 such that the equalities (10) hold, then $x \overset{O(2)}{\sim} y$. Moreover, the unique $F \in O(2)$ exists such that $y(t) = Fx(t)$ for all $t \in T$. In this case, the following cases exist:*

(a) $[x(t)\, x'(t)] > 0$ *and* $[y(t)\, y'(t)] > 0$ *for all $t \in T$;*

(b) $[x(t)\, x'(t)] > 0$ *and* $[y(t)\, y'(t)] < 0$ *for all $t \in T$;*

(c) $[x(t)\, x'(t)] < 0$ *and* $[y(t)\, y'(t)] > 0$ *for all $t \in T$;*

(d) $[x(t)\, x'(t)] < 0$ *and* $[y(t)\, y'(t)] < 0$ *for all $t \in T$.*
In cases (a) and (d), $F = U_1$, where $U_1 \in O^+(2)$ and the matrix U_1 has the form (4).
In cases (b) and (c), $F = (U_2 W)$, where $U_2 \in O^+(2)$ and the matrix U_2 has the form (8).

Proof. (i) Let $x \overset{O(2)}{\sim} y$. The functions $< x(t), x(t) >$ and $[x(t)x'(t)]^2$ are $O(2)$-invariant for all $t \in T$. So equalities (10) hold.

(ii) Conversely, assume that equalities (10) hold. Since $x(t)$ and $y(t)$ are L-non-degenerate T-paths, $[x(t)\, x'(t)] \neq 0$ and $[y(t)\, y'(t)] \neq 0$ for all $t \in T$. Hence there exist the following cases:

(a) $[x(t)\, x'(t)] > 0$ and $[y(t)\, y'(t)] > 0$ for all $t \in T$;

(b) $[x(t)\, x'(t)] > 0$ and $[y(t)\, y'(t)] < 0$ for all $t \in T$;

(c) $[x(t)\, x'(t)] < 0$ and $[y(t)\, y'(t)] > 0$ for all $t \in T$;

(d) $[x(t) \, x'(t)] < 0$ and $[y(t) \, y'(t)] < 0$ for all $t \in T$.

By the equalities (10), we have the equality $[x(t) \, x'(t)]^2 = [y(t) \, y'(t)]^2$. In cases (a) and (d) this equality implies $[x(t) \, x'(t)] = [y(t) \, y'(t)]$. Then, by Theorem 3.2, we obtain that there exists the unique $F \in O^+(2)$ such that $y(t) = Fx(t)$ for all $t \in T$ and $Fx(t) = U_1 x(t)$, where the matrix U_1 has the form (4).

In cases (b) and (c), the equality $[x(t) \, x'(t)]^2 = [y(t) \, y'(t)]^2$ implies $[x(t) \, x'(t)] = -[y(t) \, y'(t)]$ for all $t \in T$. Using this equality and $det(W) = -1$, we obtain

$$[Wx(t) \, (Wx)'(t)] = det(W) \, [x(t) \, x'(t)] = -[x(t) \, x'(t)] = [y(t) \, y'(t)]$$

for all $t \in T$. This equality and the equality
$< Wx(t), Wx(t)) = < x(t), x(t) > = < y(t), y(t) >$ imply the equalities (3) for paths $Wx(t)$ and $y(t)$. By Theorem 3.2(ii), the unique $U_2 \in O^+(2)$ exists such that $y(t) = U_2(Wx(t))$, where U_2 has the form (8). Since $U_2 \in O^+(2)$, we have $U_2 W \in O^+(2)W$. $\qquad \square$

Lemma 3.8. *For all vectors y_1, y_2, z_1, z_2 in E_2, the equality $[y_1 y_2][z_1 z_2] = < y_1, z_1 > < y_2, z_2 > - < y_1, z_2 > < y_2, z_1 >$ holds.*

Proof. A proof of the this lemma is given in ([16, Lemma 13]). $\qquad \square$

Theorem 3.9. *(i) Let $x(t), y(t)$ be L-non-degenerate T-paths in E_2 such that $x \overset{O(2)}{\sim} y$. Then, for all $t \in T$, the following equalities hold*

$$\begin{cases} < x(t), x(t) > = < y(t), y(t) >, \\ < x'(t), x'(t) > = < y'(t), y'(t) > . \end{cases} \tag{11}$$

(ii) Conversely, if $x(t)$ and $y(t)$ are L-non-degenerate T-paths in E_2 such that the equalities (11) hold, then $x \overset{O(2)}{\sim} y$. Moreover, the unique $F \in O(2)$ exists such that $y(t) = Fx(t)$ for all $t \in T$. In this case, only cases $(a), (b), (c), (d)$ in Theorem 3.7 exist.
In cases (a) and (d), $F = U_1$, where $U_1 \in O^+(2)$ and U_1 has the form (4).
In cases (b) and (c), $F = (U_2 W)$, where $U_2 \in O^+(2)$ and U_2 has the form (8).

Proof. (i) Let $x \overset{O(2)}{\sim} y$. Since functions $< x(t), x(t) >$ and $< x'(t), x'(t) >$ are $O(2)$-invariants, so the equalities (11) hold.

(ii) Conversely, assume that the equalities (11) hold.

Applying Lemma 3.8 to vectors $y_1 = z_1 = x(t), y_2 = z_2 = x'(t)$, we obtain the following equality

$$[x(t)\, x'(t)]^2 = < x(t), x(t) > < x'(t), x'(t) > - < x(t), x'(t) >^2 \qquad (12)$$

for all $t \in T$.

As in the proof of Theorem 3.2, the equality $< x(t), x(t) >=< y(t), y(t) >$ for all $t \in T$ implies the equality $< x(t), x'(t) >=< y(t), y'(t) >$ for all $t \in T$. Using the equalities (11), the equality $< x(t), x'(t) >=< y(t), y'(t) >$ and the equality (12) for all $t \in T$, we obtain the equalities (10). Then the proof of the present theorem follows from Theorem 3.7. $\qquad\square$

4   Theorems on an existence for L-regular paths

Theorem 4.1.   *(i) Let $a(t)$ be a real $C^{(1)}$-function on T such that $a(t) > 0$ for all $t \in T$, and $b(t)$ be a real continuous function on T. Then there exists an L-regular T-path $x(t)$ in E_2 such that*

$$\begin{cases} < x(t), x(t) >= a(t), \\ \quad [x(t)\, x'(t)] = b(t) \end{cases} \qquad (13)$$

for all $t \in T$. Moreover, a general solution of this system of equations has the following form

$$x(t) = e^{i\varphi} \sqrt{a(t)} e^{i \int_{t_0}^t \frac{b(u)}{a(u)} du}, \qquad (14)$$

where $\varphi \in [0, 2\pi)$ and $t_0 \in T$.

(ii) Conversely, every path of the form (14) in E_2 is an L-regular path such that the equalities (13) hold.

Proof.   (i) Assume that an L-regular path $x(t)$ in E_2 such that the equalities (13) hold. The identity $vu^{-1} = \frac{<u,v>}{Q(u)} + i\frac{[u\,v]}{Q(u)}$ in Proposition 3.1 implies the following identity for the L-regular path $x(t)$:

$$x'(t)(x(t))^{-1} = \frac{< x(t), x'(t) >}{Q(x(t))} + i\frac{[x(t)\, x'(t)]}{Q(x(t))}.$$

This identity, (13) and the equality $< x(t), x'(t) >= \frac{1}{2}\frac{d}{dt} < x(t), x(t) >$ imply the equality:

$$\frac{x'(t)}{x(t)} = \frac{1}{2}\frac{a'(t)}{a(t)} + i\frac{b(t)}{a(t)}.$$

Then we obtain the following differential equation for the path $x(t)$:

$$x'(t) = (\frac{1}{2}\frac{a'(t)}{a(t)} + i\frac{b(t)}{a(t)})x(t).$$

This differential equation has the following general solution:

$$x(t) = Ke^{\int_{t_0}^t (\frac{1}{2}\frac{a'(u)}{a(u)} + i\frac{b(u)}{a(u)})du},$$

where $K \in \Omega$. Since $x(t)$ is an L-regular path, we have $x(t) \neq 0$ for all $t \in T$. This inequality and the last above equality imply that $K \neq 0$. Using equalities $< x(t), x(t) > = x(t)\overline{x(t)}$ and $< x(t), x(t) > = a(t)$, we obtain the following equality

$$a(t) = < x(t), x(t) > = < Ke^{\int_{t_0}^t (\frac{1}{2}\frac{a'(u)}{a(u)} + i\frac{b(u)}{a(u)})du}, Ke^{\int_{t_0}^t (\frac{1}{2}\frac{a'(u)}{a(u)} + i\frac{b(u)}{a(u)})du} >$$

$$= Ke^{\int_{t_0}^t (\frac{1}{2}\frac{a'(u)}{a(u)} + i\frac{b(u)}{a(u)})du}\overline{K}e^{\int_{t_0}^t (\frac{1}{2}\frac{a'(u)}{a(u)} - i\frac{b(u)}{a(u)})du}$$

$$= |K|^2 e^{\int_{t_0}^t (\frac{1}{2}\frac{a'(u)}{a(u)})du}e^{\int_{t_0}^t (\frac{1}{2}\frac{a'(u)}{a(u)})du}.$$

This equality implies the equality $|K| e^{\int_{t_0}^t (\frac{1}{2}\frac{a'(u)}{a(u)})du} = \sqrt{a(t)}$. Using this equality and the equality $K = |K| e^{i\varphi}$, where $0 \leq \varphi < 2\pi$, we obtain

$$x(t) = Ke^{\int_{t_0}^t (\frac{1}{2}\frac{a'(u)}{a(u)} + i\frac{b(u)}{a(u)})du} = |K| e^{\int_{t_0}^t (\frac{1}{2}\frac{a'(u)}{a(u)})du}e^{i\varphi}e^{\int_{t_0}^t (i\frac{b(u)}{a(u)})du}$$

$$= e^{i\varphi}\sqrt{a(t)}e^{\int_{t_0}^t (i\frac{b(u)}{a(u)})du}.$$

This equality implies the form (14).

(ii) Conversely, assume that an L-regular path $x(t)$ has the form (14). Then, we have $x(t) = e^{i\varphi}\sqrt{a(t)}e^{i\int_{t_0}^t \frac{b(u)}{a(u)}du}$. This equality implies

$$< x(t), x(t) > = < e^{i\varphi}\sqrt{a(t)}e^{i\int_{t_0}^t \frac{b(u)}{a(u)}du}, e^{i\varphi}\sqrt{a(t)}e^{i\int_{t_0}^t \frac{b(u)}{a(u)}du} > =$$

$$e^{i\varphi}\sqrt{a(t)}e^{i\int_{t_0}^t \frac{b(u)}{a(u)}du}e^{-i\varphi}\sqrt{a(t)}e^{-i\int_{t_0}^t \frac{b(u)}{a(u)}du} = a(t).$$

Hence $< x(t), x(t) > = a(t)$. The equality $x(t) = e^{i\varphi}\sqrt{a(t)}e^{i\int_{t_0}^t \frac{b(u)}{a(u)}du}$ implies

$$x'(t) = e^{i\varphi}\frac{1}{2}\frac{a'(t)}{\sqrt{a(t)}}e^{i\int_{t_0}^t \frac{b(u)}{a(u)}du} + e^{i\varphi}\sqrt{a(t)}e^{i\int_{t_0}^t \frac{b(u)}{a(u)}du}(i\frac{b(t)}{a(t)}).$$

This equality and the equality $x(t) = e^{i\varphi}\sqrt{a(t)}e^{i\int_{t_0}^{t}\frac{b(u)}{a(u)}du}$ imply $[x(t)\, x'(t)] =$

$$\left[e^{i\varphi}\sqrt{a(t)}e^{i\int_{t_0}^{t}\frac{b(u)}{a(u)}du}\quad (e^{i\varphi}\frac{1}{2}\frac{a'(t)}{\sqrt{a(t)}}e^{i\int_{t_0}^{t}\frac{b(u)}{a(u)}du}+e^{i\varphi}\sqrt{a(t)}e^{i\int_{t_0}^{t}\frac{b(u)}{a(u)}du}(i\frac{b(t)}{a(t)}))\right]=$$

$$\left[e^{i\varphi}\sqrt{a(t)}e^{i\int_{t_0}^{t}\frac{b(u)}{a(u)}du}\quad e^{i\varphi}\frac{1}{2}\frac{a'(t)}{\sqrt{a(t)}}e^{i\int_{t_0}^{t}\frac{b(u)}{a(u)}du}\right]+$$

$$\left[e^{i\varphi}\sqrt{a(t)}e^{i\int_{t_0}^{t}\frac{b(u)}{a(u)}du}\quad e^{i\varphi}\sqrt{a(t)}e^{i\int_{t_0}^{t}\frac{b(u)}{a(u)}du}(i\frac{b(t)}{a(t)})\right]=$$

$$\frac{1}{2}a'(t)\left[e^{i\varphi}e^{i\int_{t_0}^{t}\frac{b(u)}{a(u)}du}\quad e^{i\varphi}e^{i\int_{t_0}^{t}\frac{b(u)}{a(u)}du}\right]+b(t)\left[e^{i\varphi}e^{i\int_{t_0}^{t}\frac{b(u)}{a(u)}du}\quad ie^{i\varphi}e^{i\int_{t_0}^{t}\frac{b(u)}{a(u)}du}\right]=$$

$$0+b(t)\left[e^{i\varphi}e^{i\int_{t_0}^{t}\frac{b(u)}{a(u)}du}\quad ie^{i\varphi}e^{i\int_{t_0}^{t}\frac{b(u)}{a(u)}du}\right]=b(t)\left[e^{i(\varphi+\int_{t_0}^{t}\frac{b(u)}{a(u)}du)}\quad ie^{i(\varphi+\int_{t_0}^{t}\frac{b(u)}{a(u)}du)}\right]=$$

$$b(t)\begin{vmatrix}cos(\varphi+\int_{t_0}^{t}\frac{b(u)}{a(u)}du) & sin(\varphi+\int_{t_0}^{t}\frac{b(u)}{a(u)}du)\\ -sin(\varphi+\int_{t_0}^{t}\frac{b(u)}{a(u)}du) & cos(\varphi+\int_{t_0}^{t}\frac{b(u)}{a(u)}du)\end{vmatrix}=b(t).$$

Hence $[x(t)\, x'(t)] = b(t)$ and the equalities (13) hold.

$\square$

Example 4.2. Let $T = R$. Consider the following real continuous functions on T: $a(t) = \frac{4}{1+4t^2}$ and $b(t) = \frac{-8}{(1+4t^2)^2}$. In this case, the system of the equalities (13) has the following general solution:

$$x(t) = e^{i\varphi}\sqrt{a(t)}e^{i\int_{t_0}^{t}\frac{b(u)}{a(u)}du} = e^{i\varphi}\frac{2}{\sqrt{1+4t^2}}e^{i\int_{0}^{t}\frac{-2}{1+4u^2}du}$$

$$= e^{i\varphi}\frac{2}{\sqrt{1+4t^2}}e^{-iarctan(2t)}$$

$$= e^{i\varphi}\frac{2}{\sqrt{1+4t^2}}(\frac{1}{\sqrt{1+4t^2}}-i\frac{2t}{\sqrt{1+4t^2}})$$

$$= e^{i\varphi}(\frac{2}{1+4t^2}-i\frac{4t}{1+4t^2}),$$

where $\varphi \in [0, 2\pi)$.

Theorem 4.3.   (i) Let $a(t)$ be a real $C^{(1)}$-function on T such that $a(t) > 0$ for all $t \in T$. Then there exists a completely L-degenerate L-regular T-path $x(t)$ in E_2 such that

$$< x(t), x(t) >= a(t)$$

for all $t \in T$. Moreover, a general solution of this equation has the following form

$$x(t) = e^{i\varphi}\sqrt{a(t)}, \tag{15}$$

where $\varphi \in [0, 2\pi)$.

(ii) *Conversely, every T-path $x(t)$ in E_2 of the form (15) is a completely L-degenerate L-regular path.*

Proof. (i) Assume that $x(t)$ is a completely L-degenerate L-regular T-path in E_2 such that $< x(t), x(t) >= a(t)$. Since $x(t)$ is completely L-degenerate, we have $[x(t)\,x'(t)] = 0$ for all $t \in T$. Then, using Theorem 4.1 to $x(t)$, we obtain the form (15).

(ii) Conversely, assume that $x(t)$ has the form (15). We have $x(t) = e^{i\varphi}\sqrt{a(t)}$ and $x'(t) = \frac{1}{2}e^{i\varphi}\frac{a'(t)}{\sqrt{a(t)}}$. Then we obtain

$$[x(t)\,x'(t)] = \left[e^{i\varphi}\sqrt{a(t)} \quad \frac{1}{2}e^{i\varphi}\frac{a'(t)}{\sqrt{a(t)}} \right] = \frac{1}{2}a'(t)\left[e^{i\varphi}\,e^{i\varphi} \right] = 0.$$

Hence the path (15) is completely L-degenerate. It is obvious that the path (15) is L-regular. □

Theorem 4.4. (i) *For every L-non-degenerate T-path $x(t)$ in E_2, conditions $Q(x) > 0$ and $[x(t)\,x'(t)]^2 > 0$ hold for all $t \in T$.*

(ii) *Let $t_0 \in T$, $a(t)$ be a real $C^{(1)}$-function on T such that $a(t) > 0$ for all $t \in T$, and $c(t)$ be a real continuous function on T such that $c(t) > 0$ for all $t \in T$. Then there exists an L-non-degenerate T-path $x(t)$ in E_2 such that*

$$\begin{cases} < x(t), x(t) >= a(t), \\ [x(t)\,x'(t)]^2 = c(t). \end{cases}$$

for all $t \in T$. Moreover, a general solution of this system of equations has the following form

$$x(t) = e^{i\varphi}\sqrt{a(t)}e^{i\int_{t_0}^{t}\frac{\sqrt{c(u)}}{a(u)}du} \tag{16}$$

or the following form

$$x(t) = e^{i\varphi}\sqrt{a(t)}e^{-i\int_{t_0}^{t}\frac{\sqrt{c(u)}}{a(u)}du}, \tag{17}$$

where $\varphi \in [0, 2\pi)$. Conversely, T-paths of forms (16) and (17) are L-non-degenerate T-paths in E_2 such that $Q(x) = a(t)$ and $[x'(t)x^{(2)}(t)]^2 = c(t)$.

Proof. (i) It is obvious.

(ii) Assume that $x(t)$ is an L-non-degenerate T-path in E_2 such that $Q(x) = a(t)$ and $[x(t)\, x'(t)]^2 = c(t)$. Prove that $x(t)$ has the form (16) or (17). Using $c(t) > 0$ for all $t \in T$, we obtain $[x(t)\, x'(t)]^2 = (\sqrt{c(t)})^2$. This equality implies $([x(t)x'(t)] + \sqrt{c(t)})([x(t)x'(t)] - \sqrt{c(t)}) = 0$ for all $t \in T$. Since $x(t)$ is an L-non-degenerate T-path, there exists only following two cases:

(ii.1) $[x(t)\, x'(t)] > 0$ for all $t \in T$

(ii.2) $[x(t)\, x'(t)] < 0$ for all $t \in T$.
Since $\sqrt{c(t)} > 0$ for all $t \in T$, in the case $(ii.1)$, the equality $([x(t)x'(t)] + \sqrt{c(t)})([x(t)\, x'(t)] - \sqrt{c(t)}) = 0$ implies $[x(t)\, x'(t)] - \sqrt{c(t)} = 0$ for all $t \in T$. Since $\sqrt{c(t)} > 0$ for all $t \in T$, in the case $(ii.2)$, the equality $([x(t)x'(t)] + \sqrt{c(t)})([x(t)x'(t)] - \sqrt{c(t)}) = 0$ implies $[x(t)x'(t)] + \sqrt{c(t)} = 0$ for all $t \in T$. Consider the following two cases:

(j) $[x(t)\, x'(t)] - \sqrt{c(t)} = 0$ for all $t \in T$;

(jj) $[x(t)\, x'(t)] + \sqrt{c(t)} = 0$ for all $t \in T$.

In the case (j), we have $[x(t)\, x'(t)] = \sqrt{c(t)}$ and $Q(x) = a(t)$. Then, using Theorem 4.1, where $b(t) = \sqrt{c(t)}$, we obtain the path (16). As in Theorem 4.1, it is proved that the path (17) satisfies the conditions $[x(t)\, x'(t)] = \sqrt{c(t)}$ and $Q(x) = a(t)$.

In the case (jj), we have $[x(t)\, x'(t)] = -\sqrt{c(t)}$ and $Q(x) = a(t)$. Then, using Theorem 4.1, where $b(t) = -\sqrt{c(t)}$, we obtain the form (17). As in Theorem 4.1, it is proved that the path (17) satisfies the conditions $[x(t)\, x'(t)] = -\sqrt{c(t)}$ and $Q(x) = a(t)$.

$\square$

Theorem 4.5. *(i) A path $x(t)$ is an L-non-degenerate T-path in E_2 if and only if the inequality $< x(t), x(t) >< x'(t), x'(t) > - < x(t), x'(t) >^2 > 0$ holds for all $t \in T$.*

(ii) Let $t_0 \in T$, $a(t)$ be a real $C^{(1)}$-function on T such that $a(t) > 0$ for all $t \in T$ and $d(t)$ be a continuous function on T such that $a(t)d(t) - \frac{1}{4}(a'(t))^2 > 0$ for all $t \in T$. Then there exists an L-non-degenerate

T-path $x(t)$ in E_2 such that

$$\begin{cases} <x(t), x(t)> = a(t), \\ <x'(t), x'(t)> = d(t) \end{cases} \tag{18}$$

for all $t \in T$. Moreover, a general solution of this system of equations has the following form

$$x(t) = e^{i\varphi} \sqrt{a(t)} e^{i \int_{t_0}^{t} \frac{\sqrt{a(u)d(u) - \frac{1}{4}(a'(u))^2}}{a(u)} du} \tag{19}$$

or the following form

$$x(t) = e^{i\varphi} \sqrt{a(t)} e^{-i \int_{t_0}^{t} \frac{\sqrt{a(u)d(u) - \frac{1}{4}(a'(u))^2}}{a(u)} du}, \tag{20}$$

where $\varphi \in [0, 2\pi)$. Conversely, every T-paths of the forms (19) and (20) are L-non-degenerate T-paths in E_2 such that the equalities (18) hold.

Proof. (i) Using (12), we obtain that $[x(t)\, x'(t)]^2 > 0$ for all $t \in T$ if and only if $<x(t), x(t)><x'(t), x'(t)> - <x(t), x'(t)>^2 > 0$ hold for all $t \in T$.

(ii) Assume that a regular T-path $x(t)$ in E_2 such that the equalities (18) hold. Prove that $x(t)$ has the form (19) or (20).

The equality $<x(t), x(t)> = a(t)$ for all $t \in T$ implies $<x(t), x'(t)> = \frac{1}{2}a'(t)$. Put $c(t) = a(t)d(t) - \frac{1}{4}(a'(t))^2$. Using $<x(t), x(t)> = a(t)$, $<x(t), x'(t)> = \frac{1}{2}a'(t)$, $<x'(t), x'(t)> = d(t)$, the equality (12) and the equality $c(t) = a(t)d(t) - \frac{1}{4}(a'(t))^2$, we obtain that $[x(t)x'(t)]^2 = c(t) > 0$ for all $t \in T$. Hence $x(t)$ is an L-non-degenerate T-path. Then, by Theorem 4.4, $x(t)$ has the form (16) or (17), where $c(t) = a(t)d(t) - \frac{1}{4}(a'(t))^2$. This means that $x(t)$ has the form (19) or (20).

Conversely, assume that $x(t)$ is the T-path of form (19) or (20). Using the equality (12), the equality $c(t) = a(t)d(t) - \frac{1}{4}(a'(t))^2 > 0$ for all $t \in T$ and Theorem 4.4, we obtain that $x(t)$ is an L-non-degenerate T-path such that $<x(t), x(t)> = a(t)$ and $[x(t)\, x'(t)]^2 = c(t)$, where $c(t) = a(t)d(t) - \frac{1}{4}(a'(t))^2$. Using the equality (12), the equalities $c(t) = a(t)d(t) - \frac{1}{4}(a'(t))^2$, $<x(t), x'(t)> = \frac{1}{2}a'(t)$ and $<x(t), x(t)> = a(t) > 0$ for all $t \in T$, we obtain $<x'(t), x'(t)> = d(t)$. $\square$

5   The type and invariant parametrizations of an L-non-degenerate curve

Some definitions on curves are given in the section "Introduction".

Proposition 5.1. *Let α be an L-regular curve. Then every parametrization $x \in \alpha$ is an L-regular path.*

Proof. Let α be an L-regular curve. Then there exists a parametrization $x \in \alpha$ such that $x(t)$ is an L-regular J-path, where $J = (a, b) \subseteq R$. Let $y \in \alpha$ be an arbitrary parametrization and $y(u)$ is an U-path, where $U = (c, d) \subseteq R$. Then there exists a $C^{(1)}$-diffeomorphism $\varphi : U \to J$ such that $\varphi'(r) > 0$ and $y(r) = x(\varphi(r))$ for all $r \in U$. Since $x(t)$ is an L-regular J-path, $< x(t), x(t) > \neq 0$ for all $t \in J$. We have $< y(r), y(r) > = < x(\varphi(r)), x(\varphi(r)) >$. Since $< x(t), x(t) > \neq 0$ for all $t \in J$, we have $< y(r), y(r) > = < x(\varphi(r)), x(\varphi(r)) > \neq 0$ for all $r \in U$. Hence $y(r)$ is an L-regular path. $\qquad\square$

Definition 5.2. A curve α will be called L-non-degenerate if it contains an L-non-degenerate path.

Proposition 5.3. *Let α be an L-non-degenerate curve. Then every parametrization $x \in \alpha$ is an L-non-degenerate path.*

Proof. Let α be an L-non-degenerate curve. Then there exists a parametrization $x \in \alpha$ such that $x(t)$ is an L-non-degenerate J-path, where $J = (a, b) \subseteq R$. Let $y \in \alpha$ be an arbitrary parametrization and $y(u)$ is an U-path, where $U = (c, d) \subseteq R$. Then there exists a $C^{(1)}$-diffeomorphism $\varphi : U \to J$ such that $\varphi'(r) > 0$ and $y(r) = x(\varphi(r))$ for all $r \in U$. Since $x(t)$ is an L-non-degenerate J-path, $[x(t)\, x'(t)] \neq 0$ for all $t \in J$. We have

$$[y(r)\, y'(r)] = \left[x(\varphi(r))\, \frac{dx(\varphi(r))}{dr} \right] = \left[x(\varphi(r))(\varphi'(r)\, \frac{dx(\varphi(r))}{d\varphi}) \right] =$$
$$= \varphi'(r) \left[x(\varphi(r))\, \frac{dx(\varphi(r))}{d\varphi} \right].$$

This equality and the inequality $[x(t)\, x'(t)] \neq 0$ for all $t \in J$ implies the inequality $\left[x(\varphi(r))\, \frac{dx(\varphi(r))}{d\varphi} \right] \neq 0$ for all $r \in U$. Since $\varphi'(r) > 0$ for all $r \in U$ and $\left[x(\varphi(r))\, \frac{dx(\varphi(r))}{d\varphi} \right] \neq 0$ for all $r \in U$, the above equality implies $[y(r)\, y'(r)] \neq 0$ for all $r \in U$. Hence $y(r)$ is an L-non-degenerate path. $\qquad\square$

Now we define invariant parametrizations of L-non-degenerate curves in E_2. Let $x(t)$ be an L-non-degenerate J-path in E_2, where $J = (a, b) \subseteq R, a < b$.

For $c, d \in R$ such that $a < c < d < b$, we put $l_x(c, d) = \int\limits_{c}^{d} |[x(t)\, x'(t)]|\, dt$. The limits $l_x(a, d) = \lim_{c \to a} l_x(c, d) \leq +\infty$ and $l_x(c, b) = \lim_{d \to b} l_x(c, d) \leq +\infty$ exist. There are only four possibilities:

(γ_1) $0 < l_x(a, d) < +\infty$, $0 < l_x(c, b) < +\infty$;

(γ_2) $0 < l_x(a, d) < +\infty$, $l_x(c, b) = +\infty$;

(γ_3) $l_x(a, d) = +\infty$, $0 < l_x(c, b) < +\infty$;

(γ_4) $l_x(a, d) = +\infty$, $l_x(c, b) = +\infty$.

Suppose that the case (γ_1) or (γ_2) holds for some $c, d \in J$. Then $l = l_x(a, d) + l_x(c, b) - l_x(c, d)$, where $0 \leq l \leq +\infty$, does not depend on $c, d \in J$. In this case we say that x belongs to the $O(2)$-type of $(0, l)$. The cases (γ_3) and (γ_4) do not depend on c, d. In these cases, we say that x belongs to the $O(2)$-types of $(-\infty, 0)$ and $(-\infty, +\infty)$, respectively. There exist paths of all $O(2)$-types $(0, l)$, where $l < +\infty$, $(0, +\infty)$, $(-\infty, 0)$, and $(-\infty, +\infty)$. The $O(2)$-type of a path x will be denoted by $TL(x)$.

Proposition 5.4. *(i) Let x and y be L-non-degenerate paths in E_2. If $x \overset{O(2)}{\sim} y$, then $TL(x) = TL(y)$;*

(ii) If x and y are parametrizations of an L-non-degenerate curve α, then $TL(x) = TL(y)$.

Proof. A proof is easy and it is omitted. $\qquad\square$

The $O(2)$-type of a path $x \in \alpha$ will be called the $O(2)$-type of the non-degenerate curve α and denoted by $TL(\alpha)$.

Proposition 5.5. *Let α and β be non-degenerate curves in E_2. If $\alpha \overset{O(2)}{\sim} \beta$, then $TL(\alpha) = TL(\beta)$.*

Proof. A proof is easy and it is omitted. $\qquad\square$

Now we define an invariant parametrization of an L-non-degenerate curve in E_2. Let $J = (a, b)$ and $x(t)$ be an L-non-degenerate J-path in E_2. We define the $O(2)$-type arc length function $s_x(t)$ for each $O(2)$-type as follows. We put $s_x(t) = l_x(a, t)$ for the case $TL(x) = (0, l)$, where $l \leq +\infty$, and $s_x(t) = -l_x(t, b)$ for the case $TL(x) = (-\infty, 0)$.

Assume that $TL(x) = (-\infty, +\infty)$. We define the function $s_x(t)$ for this case. We choose a fixed point in every interval $J = (a, b)$ of R and denote it by a_J. Put $a_J = 0$ for $J = (-\infty, +\infty)$. For the J-path $x(t)$, we set $s_x(t) = l_x(a_J, t)$.

Since $s'_x(t) > 0$ for all $t \in J$, the inverse function of $s_x(t)$ exists. Let us denote it by $t_x(s)$. The domain of $t_x(s)$ is $TL(x)$ and $t'_x(s) > 0$ for all $s \in TL(x)$.

Proposition 5.6. *Let $U = (a, b)$ and x be an L-non-degenerate U-path in E_2 and $J = (c, d) \subseteq R$. Then:*

(i) $s_{Fx}(t) = s_x(t), \forall t \in U$ and $t_{Fx}(s) = t_x(s), \forall s \in TL(x)$, for all $F \in O(2)$;

(ii) the equalities $s_{x(\varphi)}(r) = s_x(\varphi(r)) + s_0, \forall r \in J$ and $\varphi(t_{x(\varphi)}(s + s_0)) = t_x(s), \forall s \in TL(x)$ hold for any $C^{(1)}$-diffeomorphism $\varphi : J = (c, d) \to U$ such that $\varphi'(r) > 0$ for all $r \in J$, where $s_0 = 0$ for $TL(x) \neq (-\infty, +\infty)$ and $s_0 = l_x(\varphi(a_J), a_U)$ for $TL(x) = (-\infty, +\infty)$.

Proof. It is similar to the proof of Proposition 2 in [4]. □

Let α be an L-non-degenerate curve, $x \in \alpha$. Then $x(t_x(s))$ is a parametrization of α.

Definition 5.7. (see [4, 15]) The parametrization $x(t_x(s))$ of an L-non-degenerate curve α will be called an invariant parametrization of α.

We denote the set of all invariant parametrizations of an L-non-degenerate α by $Ip(\alpha)$. Every $y \in Ip(\alpha)$ is a J-path, where $J = TL(\alpha)$.

Proposition 5.8. *Let α be an L-non-degenerate curve, $x \in \alpha$ and x be a J-path, where $J = TL(\alpha)$. Then the following conditions are equivalent:*

(i) x is an invariant parametrization of α;

(ii) $|[x(s) \, x'(s)]| = 1$ for all $s \in TL(\alpha)$;

(iii) $s_x(s) = s$ for all $s \in TL(\alpha)$.

Proof. It is similar to the proof of Proposition 3 in [4]. □

Below $J \subseteq R$, where J is a one of sets $(0, l), l < +\infty$; $(0, +\infty)$, $(-\infty, 0)$ or $(-\infty, +\infty)$.

Theorem 5.9. *Let α be an L-non-degenerate curve in E_2 and $x(s) \in Ip(\alpha)$. Then $x(s)$ has the following form*

$$x(s) = e^{i\varphi} \sqrt{a(s)} e^{i \int_{s_0}^{s} \frac{1}{a(u)} du}, \tag{21}$$

or the following form

$$x(s) = e^{i\varphi}\sqrt{a(s)}e^{-i\int_{s_0}^{s}\frac{1}{a(u)}du}, \qquad (22)$$

where $\varphi \in [0, 2\pi)$ and $a(s)$ is a real continuous functions on J.

Conversely, every path $x(s)$ of the form (21) or the form (22) is an invariant parametrization of an L-non-degenerate curve in E_2.

Proof. Let α be an L-non-degenerate curve in E_2 and $x(s) \in Ip(\alpha)$. Then, by Proposition 5.8(ii), $|[x(s)\,x'(s)]| = 1, \forall s \in TL(\alpha)$. Put $c(s) = [x(s)\,x'(s)]^2$ and $a(s) = < x(s), x(s) >$ for all $s \in TL(\alpha)$. Then $c(s) = 1$ for all $s \in TL(\alpha)$. This equality, equalities (16) and (17) imply that $x(s)$ has the form (21) in the case $[x(s)\,x'(s)] = 1, \forall s \in TL(\alpha)$ and the form (22) in the case $[x(s)\,x'(s)] = -1, \forall s \in TL(\alpha)$.

Conversely, assume that $x(s)$ has the form (21) or the form (22). Then, by Theorem 4.4, $Q(x(s)) = a(s)$ and $|[x(s)\,x'(s)]|^2 = c(s) = 1$. By Proposition 5.8(ii), the equality $|[x(s)\,x'(s)]|^2 = 1$ implies that the path $x(s)$ is an invariant parametrization. $\square$

Proposition 5.10. *Let α be an L-non-degenerate curve and $TL(\alpha) \neq (-\infty, +\infty)$. Then there exists the unique invariant parametrization of α.*

Proof. A proof is similar to the proof of Proposition 4 in [4]. $\square$

Let α be an L-non-degenerate curve and $L(\alpha) = (-\infty, +\infty)$. Then it is easy to see that the set $Ip(\alpha)$ is infinite and it is not countable.

Remark 5.11. We note that if $x(t)$ is a periodic L-non-degenerate path then $TL(x) = (-\infty, +\infty)$.

Proposition 5.12. *Let α be an L-non-degenerate curve, $TL(\alpha) = (-\infty, +\infty)$ and $x \in Ip(\alpha)$. Then $Ip(\alpha) = \{y : y(s) = x(s + c), c \in (-\infty, +\infty)\}$.*

Proof. A proof is similar to the proof of Proposition 5 in [4]. $\square$

Let H be a one of groups $O(2)$, $O^{+}(2)$.

Theorem 5.13. *Let α, β be L-non-degenerate curves and $x \in Ip(\alpha), y \in Ip(\beta)$. Then:*

(i) *for $TL(\alpha) = TL(\beta) \neq (-\infty, +\infty)$, $\alpha \overset{H}{\sim} \beta$ if and only if $x \overset{H}{\sim} y$;*

(ii) *for $TL(\alpha) = TL(\beta) = (-\infty, +\infty)$, $\alpha \overset{H}{\sim} \beta$ if and only if $x \overset{H}{\sim} y(\psi_c)$ for some $c \in (-\infty, +\infty)$, where $\psi_c(s) = s + c$.*

Proof. A proof of this theorem for the group H is similar to the proof of Theorem 1 in [4] and it is omitted. □

Theorem 5.13 reduces the problem of the H-equivalence of L-non-degenerate curves for groups $O(2)$, $O^+(2)$ to that of paths only for the case $TL(\alpha) = TL(\beta) \neq (-\infty, +\infty)$.

Definition 5.14. Let $J = (-\infty, +\infty)$. J-paths $x(t)$ and $y(t)$ will be called $[H, (-\infty, +\infty)]$-equivalent, if there exist $h \in H$ and $d \in (-\infty, +\infty)$ such that $y(t) = hx(t + d)$ for all $t \in J$.

Let α, β be L-non-degenerate curves such that $TL(\alpha) = TL(\beta) = (-\infty, +\infty)$. In this case, Theorem 5.13 reduces the problem of the H-equivalence of these curves to the problem of $[H, (-\infty, +\infty)]$-equivalence of paths.

6 Complete systems of invariants of an L-non-degenerate curve and uniqueness theorems

By Theorem 5.13, G-equivalence and uniqueness problems for curves reduce to the same problems for invariant parametrizations of curves only for the case $TL(\alpha) = TL(\beta) \neq (-\infty, +\infty)$. Below we use this reduction.

Theorem 6.1. *Let α, β be L-non-degenerate curves in E_2, $TL(\alpha) \neq (-\infty, +\infty)$, $TL(\beta) \neq (-\infty, +\infty)$ and $x \in Ip(\alpha), y \in Ip(\beta)$.*

(i) *Assume that $\alpha \overset{O^+(2)}{\sim} \beta$. Then, for all $s \in TL(\alpha)$ the following equalities hold*

$$\begin{cases} TL(\alpha) = TL(\beta), \\ < x(s), x(s) > = < y(s), y(s) >, \\ Sgn\left([x(s)\, x'(s)]\right) = Sgn\left([y(s)\, y'(s)]\right). \end{cases} \tag{23}$$

(ii) *Conversely, assume that the equalities (23) hold. Then there exists the unique $F \in O^+(2)$ such that $\beta = F\alpha$. In this case, $F = U$, where $U \in O^+(2)$, U has the following form*

$$U = \begin{pmatrix} \frac{<x(s),y(s)>}{Q(x(s))} & -\frac{[x(s)\, y(s)]}{Q(x(s))} \\ \frac{[x(s)\, y(s)]}{Q(x(s))} & \frac{<x(s),y(s)>}{Q(x(s))} \end{pmatrix}, \tag{24}$$

and the matrix U does not depend on $s \in TL(\alpha)$.

Proof. (i) Let $\alpha \overset{O^+(2)}{\sim} \beta$. By Proposition 5.5, we have $TL(\alpha) = TL(\beta)$. This equality, $\alpha \overset{O^+(2)}{\sim} \beta$ and Theorem 5.13(i) imply $x \overset{O^+(2)}{\sim} y$. By Theorem 3.2, $x \overset{O^+(2)}{\sim} y$ implies the equalities (3). By the equalities (3), the equalities $< x(s), x(s) >=< y(s), y(s) >$ and $[x(s) \, x'(s)] = [y(s) \, y'(s)]$ hold for all $s \in TL(\alpha)$. The equality $[x(s) \, x'(s)] = [y(s) \, y'(s)]$ implies the equality
$Sgn\left([x(s) \, x'(s)]\right) = Sgn\left([y(s) \, y'(s)]\right)$ for all $s \in TL(\alpha)$. Hence equalities (23) hold.

(ii) Conversely, assume that the equalities (23) hold. Since $x(s)$ and $y(s)$ are invariant parametrizations, by Proposition 5.8(ii), we have $|[x(s) \, x'(s)]| = 1$ and $|[y(s) \, y'(s)]| = 1$ for all $s \in TL(\alpha)$. These equalities imply $|[x(s) \, x'(s)]| = |[y(s) \, y'(s)]|$ for all $s \in TL(\alpha)$. This equality and the equality $Sgn\left([x(s) \, x'(s)]\right) = Sgn\left([y(s) \, y'(s)]\right)$ imply the equality $[x(s) \, x'(s)] = [y(s) \, y'(s)]$ for all $s \in TL(\alpha)$. This equality and the equalities (23) imply the equalities (3). By Theorem 3.2, these equalities imply $x \overset{O^+(2)}{\sim} y$ and there exists the unique $F \in O^+(2)$ such that $y(s) = Fx(s)$. In this case, $F = U$, where $U \in O^+(2)$, U has the form (24). Here, the matrix U does not depend on $s \in TL(\alpha)$. Using $x \in Ip(\alpha), y \in Ip(\beta)$, Theorem 5.13($i$) and $y(s) = Fx(s)$ for all $s \in TL(\alpha)$, we obtain that $\beta = F\alpha$. $\qquad \square$

Remark 6.2. By Theorem 6.1, the system
$\{TL(\alpha), < x(s), x'(s) >, Sgn\left([x(s) \, x'(s)]\right)\}$, where $x(s)$ is the invariant parametrization of the curve α, is a complete system of O^+-invariants of a curve α for the case $TL(\alpha) \neq (-\infty, +\infty)$. But they are not $O^+(2)$-invariants of a curve α, in the case $TL(\alpha) = (-\infty, +\infty)$.

Theorem 6.3. *Let α, β be L-non-degenerate curves in E_2, $TL(\alpha) = TL(\beta) = (-\infty, +\infty)$ and $x \in Ip(\alpha), y \in Ip(\beta)$.*

(i) *Assume that $\alpha \overset{O^+(2)}{\sim} \beta$. Then there exists $s_1 \in (-\infty, +\infty)$ such that the following equalities hold for all $s \in (-\infty, +\infty)$*

$$\begin{cases} < x(s + s_1), x'(s + s_1) >= < y(s), y'(s) >, \\ \qquad Sgn\left([x(s) \, x'(s)]\right) = Sgn\left([y'(s) \, y'(s)]\right). \end{cases} \qquad (25)$$

(ii) *Conversely, assume that the equalities (25) hold for some $s_1 \in (-\infty, +\infty)$ and all $s \in (-\infty, +\infty)$. Then there exists the unique $F \in O^+(2)$ such*

that $\beta = F\alpha$. In this case, $F = U$, where $U \in O^+(2)$, U has the following form

$$U = \begin{pmatrix} \frac{<x(s+s_1),y(s)>}{Q(x(s+s_1))} & -\frac{[x(s+s_1)\ y(s)]}{Q(x(s+s_1))} \\ \frac{[x(s+s_1\ )y(s)]}{Q(x(s+s_1))} & \frac{<x(s+s_1),y(s)>}{Q(x(s+s_1))} \end{pmatrix}, \tag{26}$$

and the matrix U does not depend on $s \in TL(\alpha)$.

Proof. (i) Let $\alpha \overset{O^+(2)}{\sim} \beta$. By the supposition in this theorem, we have $TL(\alpha) = TL(\beta) = (-\infty, +\infty)$. By this equality, $\alpha \overset{O^+(2)}{\sim} \beta$ and Theorem 5.13(ii), there exists $s_1 \in (-\infty, +\infty)$ such that $x(s + s_1) \overset{O^+(2)}{\sim} y(s)$. By Theorem 3.2(i), $x(s + s_1) \overset{O^+(2)}{\sim} y(s)$ implies the equalities (3) for $x(s + s_1)$ and $y(s)$. That is,

$$\begin{cases} < x(s + s_1), x'(s + s_1) >= < y(s), y'(s) >, \\ \ [x(s + s_1)\ x'(s + s_1)] = [y(s)\ y'(s)]\,. \end{cases}$$

Hence the equality $< x(s + s_1), x'(s + s_1) >=< y(s), y'(s) >$ holds for all $s \in TL(\alpha)$. The equality $[x(s + s_1)\ x'(s + s_1)] = [y(s)\ y'(s)]$ implies the equality
$Sgn\,([x(s)\ x'(s)]) = Sgn\,([x(s + s_1)\ x'(s + s_1)]) = Sgn\,([y(s)\ y'(s)])$ for all $s \in TL(\alpha)$. Hence the equalities (25) hold.

(ii) Conversely, assume that the equalities (25) hold for some $s_1 \in (-\infty, +\infty)$ and all $s \in (-\infty, +\infty)$. Since $x(s), y(s)$ are invariant parametrizations, by Proposition 5.8(ii), we have $|[x(s)\ x'(s)]| = 1$ and $|[y(s)\ y'(s)]| = 1$ for all $s \in TL(\alpha)$. These equalities imply $|[x(s + s_1)\ x'(s + s_1)]| = |[x(s)\ x'(s)]| = |[y(s)\ y'(s)]|$ for all $s \in TL(\alpha)$. These equalities and $Sgn\,([x(s)\ x'(s)]) = Sgn\,([y(s)\ y'(s)])$ imply the equality $[x(s + s_1)\ x'(s + s_1)] = [y(s)\ y'(s)]$ for all $s \in TL(\alpha)$. This equality and the equalities (25) imply the equalities

$$\begin{cases} < x(s + s_1), x'(s + s_1) >= < y(s), y'(s) >, \\ \ [x(s + s_1)\ x'(s + s_1)] = [y(s)\ y'(s)] \end{cases}$$

for all $s \in TL(\alpha)$. By Theorem 3.2, $x(s + s_1) \overset{O^+(2)}{\sim} y(s)$ and there exists the unique $F \in O^+(2)$ such that $y(s) = Fx(s + s_1)$ for all $s \in TL(\alpha)$. In this case, $F = U$, where $U \in O^+(2)$, U has the form (26). Here, the matrix U does not depend on $s \in TL(\alpha)$. Using $x \in Ip(\alpha), y \in Ip(\beta)$,

Theorem 5.13(ii) and $y(s) = Fx(s + s_1)$ for all $s \in TL(\alpha)$, we obtain that $\beta = F\alpha$.

$\square$

Theorem 6.4. *Let α, β be L-non-degenerate curves in E_2, $TL(\alpha) \neq (-\infty, +\infty)$, $TL(\beta) \neq (-\infty, +\infty)$ and $x \in Ip(\alpha), y \in Ip(\beta)$.*

(i) Assume that $\alpha \overset{O(2)}{\sim} \beta$. Then we have

$$\begin{cases} TL(\alpha) = TL(\beta), \\ <x(s), x'(s)> = <y(s), y'(s)> \end{cases} \tag{27}$$

for all $s \in TL(\alpha)$.

(ii) Conversely, assume that the equalities (27) hold. Then $\alpha \overset{O(2)}{\sim} \beta$. Moreover, the unique $F \in O(2)$ exists such that $\beta = F\alpha$. In this case, only two following cases exist:

(ii.1) $Sgn\left([x(s)\, x'(s)]\right) = Sgn\left([y(s)\, y'(s)]\right)$;

(ii.2) $Sgn\left([x(s)\, x'(s)]\right) = -Sgn\left([y(s)\, y'(s)]\right)$.

In the case (ii.1), $F = U_1$, where U_1 has the form (26).

In the case (ii.2), $F = (U_2 W)$, where U_2 has the following form

$$U_2 = \begin{pmatrix} \frac{<Wx(s), y(s)>}{Q(Wx(s))} & -\frac{[(Wx(s))\, y(s)]}{Q(Wx(s))} \\ \frac{[(Wx(s))\, y(s)]}{Q(Wx(s))} & \frac{<Wx(s), y(s)>}{Q(Wx(s))} \end{pmatrix},$$

.

Here, the matrices U_1 and U_2 do not depend on $s \in TL(\alpha)$.

Proof. A proof follows easy from Theorems 5.13(i), 3.9 and 6.1. $\square$

Theorem 6.5. *Let α, β be L-non-degenerate curves in E_2, $TL(\alpha) = TL(\beta) = (-\infty, +\infty)$ and $x \in Ip(\alpha), y \in Ip(\beta)$.*

(i) Assume that $\alpha \overset{O(2)}{\sim} \beta$. Then there exists $s_1 \in (-\infty, +\infty)$ such that the following equality

$$<x(s + s_1), x'(s + s_1)> = <y(s), y'(s)> \tag{28}$$

holds for all $s \in (-\infty, +\infty)$.

(ii) *Conversely, assume that there exists $s_1 \in (-\infty, +\infty)$ such that the equality (28) holds for all $s \in (-\infty, +\infty)$. Then $\alpha \overset{O(2)}{\sim} \beta$. Moreover, $F \in O(2)$ exists such that $\beta = F\alpha$. In this case, following cases exist:*

(ii.1) $[x(s + s_1)\, x'(s + s_1)] = [y(s)\, y'(s)];$

(ii.2) $[x(s + s_1)\, x'(s + s_1)] = -[y(s)\, y'(s)].$

In the case (ii.1), $F = U_1$, where U_1 has the form (26).

In the case (ii.2), $F = (U_2 W)$, where U_2 has the following form

$$
U_2 = \begin{pmatrix} \frac{<Wx(s+s_1), y(s)>}{Q(Wx(s+s_1))} & -\frac{[Wx(s+s_1)\, y(s)]}{Q(Wx(s+s_1))} \\ \frac{[Wx(s+s_1)\, y(s)]}{Q(Wx(s+s_1))} & \frac{<Wx(s+s_1), y(s)>}{Q(Wx(s+s_1))} \end{pmatrix}.
$$

Here, the matrices U_1 and U_2 do not depend on $s \in TL(\alpha)$.

Proof. A proof follows easy from Theorems 5.13(ii), 3.9 and 6.3. $\square$

6.1 Conclusion

Methods, developed in the present work, will be useful in the theory of global invariants of curves in two-dimensional geometries and physics. This approach is also will be useful in the theory of global invariants of vector fields and a system of curves in two-dimensional geometries and physics.

7 Acknowledgements

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